

# TOPOLOGIES ON GROUPS DETERMINED BY SEQUENCES: ANSWERS TO SEVERAL QUESTIONS OF I.PROTASOV AND E.ZELENYUK

TARAS BANAKH

ABSTRACT. Answering questions of Protasov and Zelenyuk we prove the following results:

1. For every increasing function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $\lim_{n \rightarrow \infty} f(n+1) - f(n) = \infty$  and every metrizable totally bounded group topology  $\tau$  on  $\mathbb{Z}$  there exists a convergent to zero sequence  $(a_n)_{n \in \omega}$  in  $(\mathbb{Z}, \tau)$  such that  $\lim_{n \rightarrow \infty} \frac{a_n}{f(n)} = 1$ .
2. For every real  $r > 1$  there exists a sequence  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$  but there is no ring topology  $\tau$  on  $\mathbb{Z}$  such that  $(a_n)_{n \in \omega}$  converges to zero in  $(\mathbb{Z}, \tau)$ .
3. There exists a countable topological Abelian group  $G$  determined by a  $T$ -sequence and containing a closed subgroup  $H$  which is not determined by a  $T$ -sequence but is homeomorphic to  $G$ .
4. There exist two group topologies  $\tau_1, \tau_2$  determined by  $T$ -sequences on  $\mathbb{Z}$  such that the topology  $\tau_1 \vee \tau_2$  is not complete and thus is not determined by a  $T$ -sequence.
5. There exists a countable Abelian group admitting a group topology  $\tau$  determined by a  $T$ -sequence and a metrizable group topology  $\tau'$  such that the topology  $\tau \vee \tau'$  is not discrete but contains no non-trivial convergent sequence.

In this note we give answers to several problems posed by I.Protasov and E.Zelenyuk in [PZ<sub>1</sub>] and [PZ<sub>2</sub>]. Following [PZ<sub>2</sub>] we define a sequence  $(a_n)_{n \in \omega}$  of elements of a group  $G$  to be a  $T$ -sequence if  $(a_n)_{n \in \omega}$  converges to zero in some non-discrete Hausdorff group topology on  $G$ . Given a  $T$ -sequence  $(a_n)_{n \in \omega}$  in  $G$  we denote by  $(G|(a_n))$  the group  $G$  endowed with the strongest topology in which the sequence  $(a_n)$  converges to zero. We say that a topological group  $G$  is *determined by a  $T$ -sequence* if  $G = (G|(a_n))$  for some  $T$ -sequence  $(a_n)_{n \in \omega}$  in  $G$ .

---

1991 *Mathematics Subject Classification.* 22A05, 26A12, 54H11.

Research supported in part by grant INTAS-96-0753.

1. THERE IS NO RESTRICTION ON THE GROWTH OF  $T$ -SEQUENCES IN  $\mathbb{Z}$ 

All  $T$ -sequences of integers constructed in [PZ<sub>1</sub>] have exponential growth. This led I. Protasov and E. Zelenyuk to the following question (see [PZ<sub>2</sub>] and [PZ<sub>1</sub>, Question 2.2.3]): *is there a monotone  $T$ -sequence of integers having polynomial growth?* First our result answers this question affirmatively. We recall that a group topology  $\tau$  on a group  $G$  is called *totally bounded* if for every neighborhood  $U \in \tau$  of zero in  $G$  there exists a finite subset  $F \subset G$  with  $G = F \cdot U$ .

**Theorem 1.**

- (1) *If  $(a_n)_{n=1}^\infty \subset \mathbb{Z}$  is an increasing  $T$ -sequence, then  $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = \infty$ .*
- (2) *Suppose  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $\varepsilon : \mathbb{N} \rightarrow [0, \infty)$  are functions such that  $\lim_{n \rightarrow \infty} \varepsilon(n) = \infty$  and  $\lim_{n \rightarrow \infty} f(n+1) - f(n) = \infty$ . For every metrizable totally bounded group topology  $\tau$  on  $\mathbb{Z}$  there exists a converging to zero sequence  $(a_n)_{n \in \omega} \subset (\mathbb{Z}, \tau)$  such that  $\lim_{n \rightarrow \infty} \frac{a_n}{f(n)} = 1$  and  $|a_n - f(n)| \leq \varepsilon(n)$  for every  $n \in \omega$ .*

*Proof.* 1. Suppose  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  is an increasing  $T$ -sequence with  $\lim_{n \rightarrow \infty} a_{n+1} - a_n \neq \infty$ . This means that for some  $C \in \mathbb{N}$  and every  $n \in \mathbb{N}$  we can find  $m \geq n$  with  $a_{m+1} - a_m \leq C$ . Let  $\tau$  be a non-discrete Hausdorff group topology on  $\mathbb{Z}$  such that  $(a_n)_{n=1}^\infty$  converges to zero in  $\tau$ . Pick a  $\tau$ -open neighborhood  $U \subset \mathbb{Z}$  of zero such that  $U \cap (i+U) = \emptyset$  for every  $1 \leq i \leq C$  and find  $n_0 \in \mathbb{N}$  such that  $a_n \in U$  for every  $n \geq n_0$ . By the choice of the constant  $C$ , there exists  $m \geq n_0$  with  $a_{m+1} - a_m \leq C$ . Then letting  $i = a_{m+1} - a_m$ , we get  $a_{m+1} = a_m + i \in (i+U) \cap U = \emptyset$ , a contradiction.

2. Suppose functions  $f$  and  $\varepsilon$  satisfy the hypotheses of the theorem. Without loss of generality,  $\varepsilon(1) = 0$  and  $\varepsilon(n) \leq \frac{1}{2} \min\{\sqrt{f(n)}, f(n+1) - f(n), f(n) - f(n-1)\}$  for  $n > 1$ .

Let  $\tau$  be any metrizable totally bounded group topology on  $\mathbb{Z}$  and  $\mathbb{Z} = U_0 \supset U_1 \supset U_2 \supset \dots$  be a countable base of neighborhoods of zero in  $(\mathbb{Z}, \tau)$ . For every  $n \in \omega$  let  $k(n) = \max\{i \in \omega : U_i \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)] \neq \emptyset\}$  and  $a_n$  be any point in  $U_{k(n)} \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)]$  (the number  $k(n)$  is finite since the topology  $\tau$  is Hausdorff). Evidently,  $|f(n) - a_n| \leq \varepsilon(n)$  for every  $n \in \omega$  and  $0 \leq \lim_{n \rightarrow \infty} \left| \frac{a_n}{f(n)} - 1 \right| \leq \lim_{n \rightarrow \infty} \frac{\varepsilon(n)}{f(n)} \leq \lim_{n \rightarrow \infty} \frac{1}{\sqrt{f(n)}} = 0$ .

It remains to verify the convergence of the constructed sequence  $(a_n)_{n \in \omega}$  to zero in the topology  $\tau$ . This will follow as soon as we prove that  $\lim_{n \rightarrow \infty} k(n) = \infty$ . Fix any number  $m \in \mathbb{N}$ . We have to find  $n_0 \in \mathbb{N}$  such that  $k(n) \geq m$  for every  $n \geq n_0$ . Using the total boundedness of the topology  $\tau$ , find  $l \in \mathbb{N}$  such that  $\bigcup_{|i| < l} (i + U_m) = \mathbb{Z}$ . Since  $\lim_{n \rightarrow \infty} \varepsilon(n) = \infty$ , there exists  $n_0 \in \mathbb{N}$  such that  $\varepsilon(n) > l$  for all  $n \geq n_0$ . It follows that for every  $n \geq n_0$  there exists  $i \in \mathbb{Z}$  such that  $|i| < l < \varepsilon(n)$  and  $i + U_m \ni f(n)$ . Consequently,  $U_m \cap [f(n) - \varepsilon(n), f(n) + \varepsilon(n)] \neq \emptyset$  and hence  $k(n) \geq m$ .  $\square$

*Remark 1.* The requirement of the metrizability of the topology  $\tau$  in Theorem 1 is essential: according to [PZ<sub>1</sub>, §5.1], there exists a totally bounded group topology  $\tau$  on  $\mathbb{Z}$  such that the space  $(\mathbb{Z}, \tau)$  contains no nontrivial convergent sequence.

*Remark 2.* Theorem 1 gives a short proof of Theorem 2.2.6 from [PZ<sub>1</sub>] which states that for every real number  $r \geq 1$  there exists a  $T$ -sequence  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  with  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$  (apply Theorem 1 with  $f(n) = r^n + n^2$  and  $\varepsilon(n) = n$ ).

## 2. $T$ -SEQUENCES IN THE RING $\mathbb{Z}$ .

According to Theorem 2.2.3 [PZ<sub>1</sub>], if  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  is a sequence such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$  is a transcendental real number, then  $(a_n)_{n \in \omega}$  is a  $T$ -sequence in the group  $\mathbb{Z}$ . In [PZ<sub>2</sub>] (see also [PZ<sub>1</sub>, Question 3.4.1]) I. Protasov and E. Zelenyuk asked if such a sequence  $(a_n)_{n \in \omega}$  needs to be a  $T$ -sequence in the ring  $\mathbb{Z}$ , i.e., if  $(a_n)_{n \in \omega}$  converges to zero for some Hausdorff ring topology  $\tau$  on  $\mathbb{Z}$ . The following theorem answers this question in negative.

**Theorem 2.** *For every real number  $r > 1$  there exists a sequence  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$  but  $(a_n)_{n \in \omega}$  is not a  $T$ -sequence in the ring  $\mathbb{Z}$ .*

*Proof.* Given a real number  $r > 1$  consider the sequence  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  defined by

$$a_n = \begin{cases} [r^{n/2}]^2 + 1, & \text{if } n = 2 \cdot 3^k \text{ for some } k \in \mathbb{N}; \\ [r^n], & \text{otherwise,} \end{cases}$$

where as usual  $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$  for a real number  $x$ . It can be easily shown that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$ . Nonetheless, the sequence  $(a_n)_{n \in \omega}$  can not converge to zero in a ring topology  $\tau$  on  $\mathbb{Z}$  because  $a_{2 \cdot 3^k} - a_{3^k}^2 = 1$  for every  $k \in \omega$ .  $\square$

## 3. ON CLOSED SUBGROUPS OF TOPOLOGICAL GROUPS DETERMINED BY $T$ -SEQUENCES.

In this section we give an example of a countable Abelian topological group  $G$  determined by a  $T$ -sequence and a closed subgroup  $H$  of  $G$  which is not determined by a  $T$ -sequence, thus answering Question 2.3.1 of [PZ<sub>1</sub>]. The group  $G$  is a Graev free topological Abelian group  $A(S_0)$  over a convergent sequence  $S_0$  under which we understand any countable compactum  $S_0$  with a unique nonisolated point considered as the distinguished point of  $S_0$ . We recall that the Graev free topological Abelian group  $A(X)$  over a pointed Tychonov space  $(X, *)$  is uniquely determined by the following three requirements: (1) there is an embedding  $X \subset A(X)$  such that the fixed point  $*$  of  $X$  coincides with the neutral element of the group  $A(X)$ , (2)  $A(X)$  coincides with the group hull of  $X$  in  $A(X)$ , and (3) every continuous map  $f : X \rightarrow G$  into a topological Abelian group  $G$  such that  $f(*) = 0$  uniquely extends to a continuous group homomorphism  $\bar{f} : A(X) \rightarrow G$ , see [Gr].

It will be more convenient to work with the following concrete realization of a free group  $A(S_0)$ . In the group  $\mathbb{Z}^\omega$  consider the sequence  $(e_n)_{n \in \omega} \subset \mathbb{Z}^\omega$  of characteristic functions  $e_n = \chi_{\{n\}} : \omega \rightarrow \{0, 1\} \subset \mathbb{Z}$  of one-point subsets  $\{n\} \subset \omega$ . Clearly, the sequence  $(e_n)_{n \in \omega}$  converges to zero in the product topology of  $\mathbb{Z}^\omega$ . Denote by  $\mathbb{Z}_f^\omega$

the group hull of the set  $\{e_n : n \in \omega\}$  in  $\mathbb{Z}^\omega$ . Algebraically,  $\mathbb{Z}_f^\omega$  is the direct sum of countably many of cyclic groups  $\mathbb{Z}$  and consists of all eventually zero sequences of integers. It can be easily shown that a Graev free topological Abelian group  $A(S_0)$  is topologically isomorphic to the group  $(\mathbb{Z}_f^\omega | (e_n))$  determined by the  $T$ -sequence  $(e_n)_{n \in \omega}$ .

**Theorem 3.** *The topological group  $A(S_0) = (\mathbb{Z}_f^\omega | (e_n))$  contains a closed subgroup  $H$  which is not determined by a  $T$ -sequence.*

*Proof.* Fix any function  $f : \omega \rightarrow \mathbb{N}$  such that the set  $f^{-1}(n)$  is infinite for every  $n \in \mathbb{N}$  and consider the closed subgroup

$$H = \{(x_i)_{i \in \omega} \in \mathbb{Z}_f^\omega : x_i \in f(i) \cdot \mathbb{Z} \text{ for every } i \in \omega\}$$

in  $(\mathbb{Z}_f^\omega | (e_n))$ . We claim that the topology of  $H$  is not determined by a  $T$ -sequence. Suppose the contrary:  $H = (H | (b_n))$  for some sequence  $(b_n)_{n \in \omega}$  convergent to zero in the topology  $\tau$  of the group  $(\mathbb{Z}_f^\omega | (e_n))$ . By the standard arguments (see Chapter 4 of [PZ<sub>1</sub>]) it can be shown that the group  $(\mathbb{Z}_f^\omega | (e_n))$  carries the strongest topology inducing the original (product) topology on each (compact) set

$$\mathbb{Z}_n^\omega = \{(x_i)_{i \in \omega} \in \mathbb{Z}_f^\omega : \sum_{i \in \omega} |x_i| \leq n\}, \quad n \in \mathbb{N}.$$

This fact and the convergence of  $(b_n)_{n \in \omega}$  in  $(\mathbb{Z}_f^\omega | (e_n))$  imply  $\{b_n : n \in \omega\} \subset \mathbb{Z}_{n_0}^\omega \cap H$  for some  $n_0 \in \mathbb{N}$ . Observe that  $H = H' \oplus H''$ , where  $H'$  and  $H''$  are the groups hulls of the sets  $\{f(n)e_n : f(n) \leq n_0\}$  and  $\{f(n)e_n : f(n) > n_0\}$ , respectively. It follows from the inclusion  $\{b_n\}_{n \in \omega} \subset \mathbb{Z}_{n_0}^\omega \cap H$  that  $\{b_n\}_{n \in \omega} \subset H'$ . Since  $H = H' \oplus H''$  carries the strongest group topology in which the sequence  $(b_n)_{n \in \omega}$  converges to zero, we conclude that the topology of the group  $H''$  is discrete. But this is not so, because  $H''$  contains an infinite compact subset  $\{0\} \cup \{(n_0+1)e_i : f(i) = n_0+1\}$ .  $\square$

*Remark 3.* Using the Zelenyuk topological classification of countable  $k_\omega$ -group ([Ze] or [PZ<sub>1</sub>, §4.3]) it can be shown that the subgroup  $H \subset A(S_0)$  constructed in Theorem 3 is homeomorphic to  $A(S_0)$ . This shows that the property of a topological group to be determined by a  $T$ -sequence is not a topological invariant.

*Remark 4.* The pathology described in Theorem 3 can not occur in the group  $\mathbb{Z}$ : for every Hausdorff group topology  $\tau$  on  $\mathbb{Z}$  every closed non-trivial subgroup  $H$  of  $(\mathbb{Z}, \tau)$  has finite index in  $\mathbb{Z}$  and thus is open. If  $(\mathbb{Z}, \tau)$  is determined by a  $T$ -sequence, then so does the open subgroup  $H$ .

**Question.** *Suppose  $G$  is a finitely-generated Abelian topological group determined by a  $T$ -sequence. Is every closed subgroup of  $G$  determined by a  $T$ -sequence?*

4. ON SUPREMUM OF GROUP TOPOLOGIES DETERMINED BY  $T$ -SEQUENCES

In [PZ<sub>2</sub>] I. Protasov and E. Zelenyuk posed the following problem: *Suppose  $\tau_1, \tau_2$  are two group topologies on  $\mathbb{Z}$  determined by  $T$ -sequences. Is the topology  $\tau_1 \vee \tau_2$  determined by a  $T$ -sequence?* We recall that  $\tau_1 \vee \tau_2$  is the weakest topology  $\tau$  on  $\mathbb{Z}$  such that the identity maps  $(\mathbb{Z}, \tau) \rightarrow (\mathbb{Z}, \tau_i)$ ,  $i = 1, 2$ , are continuous. Clearly, the group  $(\mathbb{Z}, \tau_1 \vee \tau_2)$  may be identified with the diagonal of the product  $(\mathbb{Z} \times \mathbb{Z}, \tau_1 \times \tau_2)$ .

It turns out that the supremum  $\tau_1 \vee \tau_2$  of two topologies determined by  $T$ -sequences on a countable Abelian group  $G$  may be very wild: it needs not be a  $k_\omega$ -topology as well as may be a  $k_\omega$ -topology but not determined by a  $T$ -sequence, etc.

We remind that a topological group  $G$  is called a  $k_\omega$ -group if  $G$  admits a cover  $\mathcal{K}$  by compact subspaces such that a subset  $U \subset G$  is open in  $G$  if and only if  $U \cap K$  is open in  $K$  for every  $K \in \mathcal{K}$ . According to [PZ<sub>1</sub>, Corollary 4.1.5] every countable group  $G$  determined by a  $T$ -sequence is a  $k_\omega$ -group.

Given an Abelian group  $G$  and a subgroup  $H$  of  $G$  let  $G \oplus_H G$  denote the quotient group of  $G \oplus G$  by the subgroup  $\Gamma = \{(h, -h) : h \in H\} \subset G \oplus G$ . The following result was suggested by I.V. Protasov.

**Theorem 4.** *For every subgroup  $H$  of a topological group  $G$  determined by a  $T$ -sequence there exist group topologies  $\tau_1, \tau_2$  on  $G \oplus_H G$  determined by  $T$ -sequences such that the topological group  $(G \oplus_H G, \tau_1 \vee \tau_2)$  contains an open subgroup topologically isomorphic to the group  $H$ .*

*Proof.* Let  $(a_n)_{n \in \omega} \subset G$  be a  $T$ -sequence determining the topology of the group  $G$ . Denote by  $\pi : G \oplus G \rightarrow G \oplus_H G$  the quotient homomorphism and by  $e_1, e_2 : G \rightarrow G \oplus_H G$  the injective group homomorphisms defined by  $e_1(g) = \pi(g, 0) = (g, 0) + \Gamma$  and  $e_2(g) = \pi(0, g) = (0, g) + \Gamma$  for  $g \in G$ . Observe that  $e_1(h) = (h, 0) + \Gamma = (0, h) + \Gamma = e_2(h)$  for any  $h \in H$  which allows us to define the injective homomorphism  $e : H \rightarrow G \oplus_H G$  by  $e = e_1|_H = e_2|_H$ . It is easy to see that  $e(H) = e_1(G) \cap e_2(G)$ . Using Theorem 2.1.4 of [PZ<sub>1</sub>] (or the complementability of the groups  $e_i(G)$  in  $G \oplus_H G$ ) one may show that for  $i = 1, 2$  the sequence  $(e_i(a_n))_{n \in \omega}$  is a  $T$ -sequence in  $G \oplus_H G$  determining the non-discrete topology  $\tau_i$  on  $G \oplus_H G$ . It follows that the map  $e_i : G \rightarrow (G \oplus_H G, \tau_i)$  is an open embedding. Then the map  $e : H \rightarrow (G \oplus_H G, \tau_1 \vee \tau_2)$  is a topological embedding and  $e(H) = e_1(G) \cap e_2(G)$  is an open subgroup of  $(G \oplus_H G, \tau_1 \vee \tau_2)$  isomorphic to  $H$ .  $\square$

Theorem 4 shows that the supremum  $\tau_1 \vee \tau_2$  of two group topologies determined by  $T$ -sequences may be as bad as bad are subgroups of topological groups determined by  $T$ -sequences. In particular, Theorems 3 and 4 imply

**Corollary 1.** *There exists a countable Abelian group  $G$  and two topologies  $\tau_1, \tau_2$  on  $G$  determined by  $T$ -sequences such that  $(G, \tau_1 \vee \tau_2)$  is a  $k_\omega$ -group not determined by a  $T$ -sequence.*

Theorem 4 also yields the existence of a countable Abelian group  $G$  and two topologies  $\tau_1, \tau_2$  on  $G$  determined by  $T$ -sequences such that  $(G, \tau_1 \vee \tau_2)$  is not complete and thus is not a  $k_\omega$ -group. We shall show that this fact is valid even for the group  $G = \mathbb{Z}$  thus answering Question 3 of [PZ<sub>2</sub>] (we do not know if Corollary 1 is true for the group  $G = \mathbb{Z}$ ).

It follows from Theorems 2.2.3 and 2.2.1 of [PZ<sub>1</sub>] (see also Exercise 2.2.5 in [PZ<sub>1</sub>]) that a sequence  $(a_n)_{n \in \omega} \subset \mathbb{Z}$  is a  $T$ -sequence in  $\mathbb{Z}$  provided  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$  either is infinite or is a transcendental real number. This implies that for any such a  $T$ -sequence  $(a_n)_{n \in \omega}$  and any non-zero integer  $c$  the sequence  $(a_n + c)_{n \in \omega}$  is a  $T$ -sequence too. Denote by  $\tau_1, \tau_2$  the group topologies on  $\mathbb{Z}$  determined by the  $T$ -sequences  $(a_n)_{n \in \omega}, (a_n + c)_{n \in \omega}$ . The following theorem implies that the topology  $\tau_1 \vee \tau_2$  on  $\mathbb{Z}$  is not complete and thus is not determined by a  $T$ -sequence.

**Theorem 5.** *Let  $\tau_1$  and  $\tau_2$  be the group topologies on a countable group  $G$  determined by  $T$ -sequences  $(a_n)_{n \in \omega}$  and  $(b_n)_{n \in \omega}$ , respectively. If for some non-zero element  $g \in G$  and every  $n_0 \in \omega$  there exist  $n, m \geq n_0$  with  $g = a_n^{-1}b_m$ , then the topological group  $(G, \tau_1 \vee \tau_2)$  is not complete and thus is not determined by a  $T$ -sequence.*

*Proof.* To prove the theorem it suffices to verify that the diagonal of the square  $G \times G$  (which is identified with  $(G, \tau_1 \vee \tau_2)$ ) is not closed in the product topology  $\tau_1 \times \tau_2$ . Suppose the contrary:  $G$  is closed in  $G \times G$ . Then for the point  $(g, 0) \in G \times G$  beyond the diagonal we may find two neighborhoods  $U_i \in \tau_i, i = 1, 2$ , of zero in  $G$  such that  $((U_1 g) \times U_2) \cap G = \emptyset$ . Since the sequences  $(a_n)_{n \in \omega}$  and  $(b_n)_{n \in \omega}$  converge to zero in the topologies  $\tau_1, \tau_2$ , we may find numbers  $n, m \in \omega$  such that  $a_n \in U_1, b_m \in U_2$ , and  $g = a_n^{-1}b_m$ . Then  $(b_m, b_m) = (a_n g, b_m) \in (U_1 g \times U_2) \cap G$ , a contradiction.  $\square$

*Remark 5.* Using Theorem 4 of [Ba], one may show that every non-closed subgroup of a countable  $k_\omega$ -group is not sequential. In particular, the group  $(G, \tau_1 \vee \tau_2)$  constructed in Theorem 5 is not sequential. Nonetheless, this group contains a non-trivial convergent sequence (this can be easily shown using the sequentiality of any countable  $k_\omega$ -group, see Exercise 4.3.1 of [PZ<sub>1</sub>]).

Finally, we prove the following theorem answering Question 2.5.5 of [PZ<sub>1</sub>].

**Theorem 6.** *There exists a countable Abelian group  $G$  admitting a group topology  $\tau$  determined by a  $T$ -sequence and a metrizable group topology  $\tau'$  such that the group  $(G, \tau \vee \tau')$  is not discrete but contains no non-trivial convergent sequence.*

*Proof.* Let  $(G, \tau) = (\mathbb{Z}_f^\omega | (e_n))$  be the Graev free topological Abelian group from Theorem 3. As we said the topology  $\tau$  is inductive with respect to the collection  $\{\mathbb{Z}_n^\omega\}_{n \in \omega}$ , where  $\mathbb{Z}_n^\omega = \{(x_i)_{i \in \omega} : \sum_{i \in \omega} |x_i| \leq n\}$  for  $n \in \omega$ .

Consider the metrizable topology  $\tau'$  on  $\mathbb{Z}_f^\omega$  generated by the base  $(U_n)_{n \in \omega}$ , where

$$U_n = \{(x_i)_{i \in \omega} \in \mathbb{Z}_f^\omega : x_i \in 2^n \cdot \mathbb{Z} \text{ for all } i \in \omega\}, \quad n \in \omega.$$

Let us show that the topology  $\tau \vee \tau'$  is not discrete. To see this, notice that for every  $n \in \omega$  and every open neighborhood  $U \in \tau$  of zero in  $\mathbb{Z}_f^\omega$  the intersection  $U_n \cap U$  is infinite (it contains the sequence  $(2^n e_i)_{i \geq n_0}$  for some  $n_0$ ). Next, assume that  $(b_n)_{n \in \omega}$  is a sequence convergent to zero in the topology  $\tau \vee \tau'$ . Since  $(b_n)_{n \in \omega}$  is convergent in  $(\mathbb{Z}_f^\omega, \tau)$  we get  $\{b_n : n \in \omega\} \subset \mathbb{Z}_{n_0}^\omega$  for some  $n_0$ . On the other hand, using the convergence of  $(b_n)_{n \in \omega}$  to zero in  $(\mathbb{Z}_f^\omega, \tau')$  we may find  $m_0 \in \mathbb{N}$  such that  $\{b_n : n \geq m_0\} \subset U_{n_0}$ . Since  $U_{n_0} \cap \mathbb{Z}_{n_0}^\omega = \{0\}$ , we conclude that  $b_n = 0$  for all  $n \geq m_0$ , i.e., the sequence  $(b_n)_{n \in \omega}$  is trivial.  $\square$

**Acknowledgement.** The author expresses his thanks to I.Protasov for valuable and stimulating discussions on the subject of the paper.

## REFERENCES

- [Ba] T.Banakh, *On topological groups containing a Fréchet-Urysohn fan*, Matem. Studii **9**:2 (1998), 149–154.
- [Gr] M.I. Graev, *Free topological groups*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **12** (1948), 279–324 (in Russian); English transl., Topology and Topological Algebra, Translations Series 1. American Mathematical Society **8** (1962), 305–364.
- [PZ<sub>1</sub>] I.Protasov, E.Zelenyuk, Mat.Studii. Monograph Series (1999), VNTL, Lviv.
- [PZ<sub>2</sub>] I.V.Protasov, E.G.Zelenyuk, *Topologies on  $\mathbb{Z}$  determined by sequences: seven open problems*, Matem. Studii **12**:1 (1999), 111.
- [Ze] E.Zelenyuk, *Group topologies determined by compacta*, Matem. Studii **5** (1995), 5–16. (in Russian)

DEPARTMENT OF MECHANICS AND MATHEMATICS, LVIV IVAN FRANKO NATIONAL UNIVERSITY, UNIVERSYTETSKA 1, LVIV, 79000, UKRAINE

*E-mail address:* `tbanakh@franko.lviv.ua`